

Metrics of constant scalar curvature on sphere bundles

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Abstract

Let G/H be a Riemannian homogeneous space. For an orthogonal representation ϕ of H on the Euclidean space \mathbb{R}^{k+1} , there corresponds the vector bundle $E = G \times_{\phi} \mathbb{R}^{k+1} \rightarrow G/H$ with fiberwise inner product. Provided that ϕ is the direct sum of at most two representations which are either trivial or irreducible, we construct metrics of constant scalar curvature on the unit sphere bundle UE of E . When G/H is the round sphere, we study the number of constant scalar curvature metrics in the conformal classes of these metrics.

Keywords The Yamabe problem · Constant scalar curvature · Spectrum of Laplacian · Riemannian submersion with totally geodesic fibers · Connection metric · Sphere bundle

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1 Introduction and results

Every closed Riemannian manifold of dimension ≥ 3 can be conformally deformed to have constant scalar curvature. This is a consequence of the affirmative answer to the Yamabe problem (cf. [20], [2]), the variational problem resolved in the mid 1980s through combined efforts of Yamabe [33], Trudinger [29], Aubin [1], and Schoen ([26], [28]). For a metric g of constant scalar curvature, it is then interesting to ask how many constant scalar curvature metrics of unit volume there are in the conformal class of g . In the present article, we construct metrics of constant scalar curvature on sphere bundles over spheres and study the number of constant scalar curvature metrics in their conformal classes.

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Throughout this article, $R(g)$ is the scalar curvature of a Riemannian metric g . Also, let $S^d(\rho) = (S^d, \rho^2 \overset{\circ}{g}_d)$ be the d -dimensional round sphere of radius ρ , where $\overset{\circ}{g}_d$ is the Riemannian metric on S^d of constant sectional curvature 1.

Previous study of O. Kobayashi ([18], [19]), R. Schoen [27], and Petean [24] (see also Jin–Li–Xu [16]) on the direct product of round spheres implies the following.

Theorem 1.1. *Consider the product metric $g(r) = \overset{\circ}{g}_m \oplus r^2 \overset{\circ}{g}_k$ on $S^m \times S^k$ for $m+k \geq 3$.*

1. *Assume $(m-1)r^2 \leq k$. If a metric g conformal to $g(r)$ has constant scalar curvature and has the same volume as $g(r)$, and if moreover the corresponding conformal factor only depends on the S^k -variable, then $g = g(r)$.*
2. *The same is true for $(k, m, 1/r)$ in place of (m, k, r) .*
3. *If $m(m-1)r^2 > l(l+k-1)(m+k-1) - k(k-1)$, then the conformal class of $g(r)$ contains at least $l+1$ metrics of constant scalar curvature $R(g(r))$ such that the corresponding conformal factors only depend on the S^k -variable.*
4. *The same is true for $(k, m, 1/r)$ in place of (m, k, r) .*

We should note that Kobayashi and Schoen succeeded in describing the space of all unit-volume metrics of constant scalar curvature in the conformal class of $g(r)$ provided $m = 1$ or $k = 1$. More precisely, if $m = 1$ or $k = 1$, then the metrics appearing in Theorem 1.1 are the only metrics of constant scalar curvature in the conformal class of $g(r)$ up to multiplicative constants and isometries.

The conformal factors in 3 (or 4) of Theorem 1.1 are radial in a sense that they are invariant under the cohomogeneity-one action $SO(k) \curvearrowright S^k$ (or $SO(m) \curvearrowright S^m$) fixing exactly two points. Henry–Petean [14] found non-radial solutions using isoparametric hypersurfaces.

Piccione et al. deal with other situations than $S^m(1) \times S^k(r)$, such as Riemannian direct products [21], Hopf fibrations [6], and homogeneous Riemannian submersions [7].

In this article, we attempt to draw a picture similar to Theorem 1.1 for non-trivial sphere bundles over spheres. Let $(B, \check{g}) = G/H$ be a Riemannian homogenous space. Here, G is a connected Lie group acting transitively on (B, \check{g}) by isometries, and H is the isotropy subgroup at a point of B . For an orthogonal representation $\phi : H \rightarrow SO(k+1)$ of H on the Euclidean space \mathbb{R}^{k+1} , there corresponds the vector bundle $E = G \times_{\phi} \mathbb{R}^{k+1} \rightarrow G/H$ with fiberwise inner product $\langle \cdot, \cdot \rangle$. Let $UE = \{s \in E \mid \langle s, s \rangle = 1\}$ be the unit sphere bundle of $(E, \langle \cdot, \cdot \rangle)$.

If ϕ is the trivial representation on \mathbb{R}^{k+1} , then $E = (G/H) \times \mathbb{R}^{k+1}$ and $UE = (G/H) \times S^k(1)$ are trivial bundles. Since the scalar curvature of \check{g} is constant, the product metric $g(1) := \check{g} \oplus \overset{\circ}{g}_k$ on UE has constant scalar curvature. Rescaling the fiber, we obtain the product metric $g(r) := \check{g} \oplus r^2 \overset{\circ}{g}_k$ on UE of constant scalar curvature $R(\check{g}) + k(k-1)/r^2$ for $r > 0$. For irreducible representations, we prove:

Theorem 1.2. *Let $(B, \check{g}) = G/H$ be a Riemannian homogenous space and $\phi : H \rightarrow SO(k+1)$ an orthogonal representation of H on \mathbb{R}^{k+1} . Assume ϕ is irreducible. Then,*

for every $r > 0$, there exists a Riemannian metric $g(r)$ on the unit sphere bundle UE of $E = G \times_{\phi} \mathbb{R}^{k+1}$ such that

- the projection $(UE, g(r)) \rightarrow (B, \check{g})$ is a Riemannian submersion with totally geodesic fibers each of which is isometric to the round sphere $S^k(r)$, and
- $g(r)$ has constant scalar curvature $R(\check{g}) + k(k-1)/r^2 - a^2r^2$.

Here, $a \geq 0$ is a real number independent of r .

For the proof, we look at the isotropy group as in Boeckx–Vanhecke [10]. Guijarro–Sadun–Walschap [12, Proposition 5.2] looked at the holonomy group and obtained the same result under an additional assumption that E admits a parallel connection.

If G/H is isometric to the round sphere, we can study the number of constant scalar curvature metrics in the conformal class of $g(r)$ as in Theorem 1.1.

Theorem 1.3. *Suppose $S^m(1) = G/H$, and let $g(r)$ be the metric in Theorem 1.2 for $m + k \geq 3$.*

1. *Assume $-(a^2/m)r^4 + (m-1)r^2 \leq k$. If a metric g conformal to $g(r)$ has constant scalar curvature and has the same volume as $g(r)$, and if moreover the gradient vector field of the corresponding conformal factor is tangent to the fibers, then $g = g(r)$.*
2. *Assume $-(a^2/k)r^2 + (k-1)/r^2 \leq m$. If a metric g conformal to $g(r)$ has constant scalar curvature and has the same volume as $g(r)$, and if moreover the corresponding conformal factor is constant along each fiber, then $g = g(r)$.*
3. *If $-a^2r^2 + k(k-1)/r^2 > l(l+m-1)(m+k-1) - m(m-1)$, then the conformal class of $g(r)$ contains at least $l+1$ metrics of constant scalar curvature $R(g(r))$ such that the corresponding conformal factors are constant along each fiber.*

The statement corresponding to (3) of Theorem 1.1 is not necessarily true for irreducible representations (cf. Sect. 4).

We can also build metrics of constant scalar curvature in the following situation.

Theorem 1.4. *Let $(B, \check{g}) = G/H$ be a Riemannian homogenous space and $\phi : H \rightarrow SO(k+1)$ an orthogonal representation of H on \mathbb{R}^{k+1} . Assume ϕ is the direct sum of at most two representations which are either trivial or irreducible.*

Then, there exist one or two one-parameter families of Riemannian metrics with constant scalar curvature on the unit sphere bundle UE of $E = G \times_{\phi} \mathbb{R}^{k+1}$ such that the projection $UE \rightarrow (B, \check{g})$ becomes Riemannian submersions with totally geodesic fibers. The (intrinsic) scalar curvatures of the typical fibers are not necessarily constant.

When $G/H = \mathrm{SU}(2)/\mathrm{U}(1) = \mathbb{C}P^1$ and ϕ is the direct sum of an irreducible and the 1-dimensional trivial representations, then we recover the constant scalar curvature metrics on Hirzebruch surfaces in [23]. We do not know if the statement of Theorem 1.4 holds for every orthogonal representation $\phi : H \rightarrow \mathrm{SO}(k+1)$. Theorem 1.4 is a consequence of Theorem 1.2 and the following

Theorem 1.5. *Let $E_i \rightarrow B$ be a real vector bundle of rank $k_i + 1 \geq 1$, $\langle \cdot, \cdot \rangle_i$ a fiberwise inner product on E_i , and ∇^i a metric connection on $(E_i, \langle \cdot, \cdot \rangle_i)$ for $i = 1, 2$. Take a Riemannian metric \check{g} on B and define the connection metric $g_i = \check{g} \oplus \check{g}_{k_i}$ on $\pi_i : UE_i \rightarrow B$ with respect to the Ehresmann connection induced by ∇^i . Assume the scalar curvatures of \check{g} , g_1 , and g_2 are constant.*

Then, there exist one or two one-parameter families of connection metrics with constant scalar curvature on the unit sphere bundle $\pi : U(E_1 \oplus E_2) \rightarrow (B, \check{g})$. The (intrinsic) scalar curvatures of the typical fibers are not necessarily constant.

Some remarks on Theorem 1.5 are in order. (1) By definition of connection metrics, the projection $U(E_1 \oplus E_2) \rightarrow (B, \check{g})$ becomes Riemannian submersions with totally geodesic fibers (see Sect. 2). (2) If neither ∇^1 nor ∇^2 is flat, there are two one-parameter families; otherwise there is only one (see Remark 3.10). (3) The typical fibers of these connection metrics can have constant scalar curvature only if the norms of O'Neill's integrability tensors for π_1 and π_2 are equal (see Remark 3.11).

This paper is organized as follows. In Sect. 2, we recall the notion of connection metrics. In Sect. 3, we construct Riemannian metrics of constant scalar curvature on sphere bundles. After proving Theorem 1.2 in Sect. 3.2, we introduce the fiberwise join of two sphere bundles and prove Theorem 1.5 in Sect. 3.3. Theorem 1.4 then follows easily (Sect. 3.4). In Sect. 4, we prove Theorem 1.3 after discussing the Yamabe problem and Riemannian submersion with totally geodesic fibers.

2 Connection metrics

Let $\pi : M \rightarrow B$ be a product bundle. $M = B \times F$ and π is the projection onto the first factor. For Riemannian metrics \check{g} and \hat{g} on B and F , respectively, we can define the product metric $g := \check{g} \oplus \hat{g}$ on the total space M . This procedure was generalized by Vilms to fiber bundles with structure group, and the resulting metrics are called connection metrics in modern terminology [13, §2.7]. We set up the notation in this section.

We recall the result of Hermann [15]. Let $\pi : (M, g) \rightarrow (B, \check{g})$ be a Riemannian submersion with totally geodesic fibers and assume g is complete. We call $VM = \ker \pi_*$ and $HM = VM^{\perp_g}$ the vertical and horizontal subbundles of TM , respectively. Let $\check{\gamma} : [0, 1] \rightarrow B$ be a path. For every choice of $x \in \pi^{-1}(\check{\gamma}(0))$, we can uniquely lift $\check{\gamma}$ to the path $\gamma : [0, 1] \rightarrow M$ starting from x so that the velocity vector field $\frac{d}{dt}\gamma$ is horizontal. Moreover, the parallel transport $\pi^{-1}(\check{\gamma}(0)) \rightarrow \pi^{-1}(\check{\gamma}(1))$ is an isometry. Therefore,

1. All fibers of π are mutually isometric. We choose a base point o of B , call $\pi^{-1}(o) = (F, \hat{g})$ the typical fiber, and denote by $\iota : F \rightarrow M$ the inclusion map.
2. We can introduce a Lie group H acting on (F, \hat{g}) isometrically so that the following holds. $\pi : M \rightarrow B$ is a fiber bundle with structure group H , and the holonomy group at o of the Ehresmann connection HM is contained in the image of the action $H \rightarrow \text{Isom}(F, \hat{g})$.

We summarize this situation as

$$H \stackrel{\text{isom}}{\curvearrowright} (F, \hat{g}) \xrightarrow{\iota} (M, g) \xrightarrow{\pi} (B, \check{g}). \quad (2.1)$$

Vilms [30] considered the converse. Let $H \curvearrowright F \xrightarrow{\iota} M \xrightarrow{\pi} B$ be a fiber bundle with structure group and $HM \subset TM$ an Ehresmann H -connection. Here, H is a (finite-dimensional) Lie group acting on F , $F = \pi^{-1}(o)$ for some $o \in B$, and the holonomy group at o of HM is contained in the image of the action $H \rightarrow \text{Diff}(F)$. Assume that there exists a Riemannian metric on F invariant under the action of H and that M is connected. For a metric \check{g} on B and an H -invariant metric \hat{g} on F , we can uniquely define a Riemannian metric $g := \check{g} \oplus_{HM} \hat{g}$ on M so that

1. $\pi : (M, g) \rightarrow (B, \check{g})$ is a Riemannian submersion and $(\ker \pi_*)^\perp g = HM$,
2. the parallel transport $\pi^{-1}(\check{\gamma}(0)) \rightarrow \pi^{-1}(\check{\gamma}(1))$ with respect to HM is an isometry for every path $\check{\gamma} : [0, 1] \rightarrow B$, and
3. the inclusion map $\iota : F \rightarrow M$ is isometric: $\hat{g} = \iota^* g$.

With the metric g , each fiber of $\pi : (M, g) \rightarrow (B, \check{g})$ is totally geodesic. If both \check{g} and \hat{g} are complete, then g is complete, and we recover the situation (2.1). We call $g = \check{g} \oplus_{HM} \hat{g}$ the connection metric of \check{g} and \hat{g} with respect to the Ehresmann connection HM . If $\pi : M = B \times F \rightarrow B$ is the product bundle, then the product metric $\check{g} \oplus \hat{g}$ agrees with the connection metric $\check{g} \oplus_{HM} \hat{g}$, where HM is the integrable Ehresmann connection. We usually write a connection metric simply as $g = \check{g} \oplus_{HM} \hat{g} = \check{g} \oplus \hat{g}$ whenever the Ehresmann connection HM is clear from the context.

In what follows, we always assume that the total space of a Riemannian submersion with totally geodesic fibers is complete and connected.

3 Construction of metrics

3.1 Integrability of connections

Let $\pi : (M, g) \rightarrow (B, \check{g})$ be a Riemannian submersion with totally geodesic fibers. The scalar curvature of g satisfies

$$R(g) = \pi^* R(\check{g}) + \hat{R} - |A|^2. \quad (3.1)$$

Here, \hat{R} is the scalar curvature of the fibers, and $|A|$ is the norm of O'Neill's integrability tensor. More precisely, $\hat{R}(x)$ is the scalar curvature at $x \in M$ of the Riemannian submanifold $\pi^{-1}(\pi(x)) \subset (M, g)$, and

$$|A|^2 = \sum_{i,j=1}^m g(A_{E_i} E_j, A_{E_i} E_j), \quad A_{E_i} E_j = \frac{1}{2} \mathcal{V}[E_i, E_j] \quad (3.2)$$

where $m = \dim B$, E_1, \dots, E_m is an orthonormal basis of $HM = (\ker \pi_*)^\perp$, and \mathcal{V} is the projection onto $VM = \ker \pi_*$. See [5, Chapter 9]. $|A| = 0$ on M if and only if the horizontal distribution HM is integrable.

Lemma 3.1. *Let $\pi : (M, g) \rightarrow (B, \check{g})$ be a Riemannian submersion with totally geodesic fibers. If \check{H} is a vector field on B and H its horizontal lift to M , then $H(\hat{R}) = 0$.*

Proof. Let γ be an integral curve of H . γ is a horizontal lift of an integral curve $\check{\gamma}$ of \check{H} . Since the parallel transport of fibers along $\check{\gamma}$ is an isometry, \hat{R} is constant along γ . Hence $H(\hat{R}) = 0$. \square

Proposition 3.2. *Let $(F, \hat{g}) \rightarrow (M, g) \xrightarrow{\pi} (B, \check{g})$ be a Riemannian submersion with totally geodesic fibers, and assume g has constant scalar curvature.*

Then, the norm $|A|$ of O'Neill tensor for π is constant if and only if both \check{g} and \hat{g} have constant scalar curvature. Moreover, the following dichotomy holds.

1. $|A|$ is constant if and only if $\check{g} \oplus c\hat{g}$ has constant scalar curvature for every real number $c > 0$.
2. $|A|$ is not constant if and only if the scalar curvature of $\check{g} \oplus c\hat{g}$ is not constant for every real number such that $c \neq 1, c > 0$.

Here, $\check{g} \oplus c\hat{g}$ is the connection metric with respect to $HM = (\ker \pi_*)^{\perp g}$.

Proof. The following observation is convenient. For a vector field \check{H} on B and its horizontal lift H to M , differentiate the both sides of (3.1) by H to get $H(R(g)) = \pi^*\check{H}(R(\check{g})) + H(\hat{R}) - H(|A|^2)$. Since $R(g)$ is constant, Lemma 3.1 implies

$$H(|A|^2) = \pi^*\check{H}(R(\check{g})). \quad (3.3)$$

Suppose $|A|$ is constant. Then, (3.3) implies $\check{H}(R(\check{g})) = 0$ for every vector field \check{H} on B . Hence $R(\check{g})$ is constant, and $\hat{R} = R(g) - \pi^*R(\check{g}) + |A|^2$ is also constant. The opposite implication is immediate from (3.1).

For a real number $c > 0$, (3.1) implies that the scalar curvature of $g_c := \check{g} \oplus c\hat{g}$ is

$$R(g_c) = \pi^*R(\check{g}) + c^{-1}\hat{R} - c|A|^2. \quad (3.4)$$

Suppose $|A|$ is constant. Since both $R(\check{g})$ and \hat{R} are constant, $R(g_c)$ is constant for every $c > 0$. Lastly, we claim:

$$\text{If } R(g_c) \text{ is constant for some } c \neq 1, \text{ then } |A| \text{ is constant.} \quad (3.5)$$

Subtracting $R(g)$ from $R(g_c)$, we obtain

$$R(g_c) - R(g) = (c^{-1} - 1)\hat{R} - (c - 1)|A|^2. \quad (3.6)$$

Let \check{H} be a vector field on B and H its horizontal lift to M . Since both $R(g_c)$ and $R(g)$ are constant, differentiation of the both sides of (3.6) by H yields $0 = -(c - 1)H(|A|^2)$. The assumption $c \neq 1$ and (3.3) imply $0 = H(|A|^2) = \pi^*\check{H}(R(\check{g}))$. Since \check{H} is arbitrary, $R(\check{g})$ is constant. Multiply the both sides of (3.1) by c and subtract the resulting equation from (3.4) to get

$$R(g_c) - cR(g) = (1 - c)\pi^*R(\check{g}) + (c^{-1} - c)\hat{R}.$$

\hat{R} is constant since $c^{-1} - c \neq 0$. This proves (3.5). \square

Let $\pi : (M, g) \rightarrow (B, \check{g})$ be a Riemannian submersion. If $\Psi \in \text{Isom}(M, g)$ and $\psi \in \text{Isom}(B, \check{g})$ are isometries satisfying $\pi \circ \Psi = \psi \circ \pi$, then we call the pair (Ψ, ψ) an automorphism of π .

Lemma 3.3. *Let $\pi : (M, g) \rightarrow (B, \check{g})$ be a Riemannian submersion. We denote by $\mathcal{V}, \mathcal{H} : TM \rightarrow TM$ the projections onto VM, HM , respectively.*

If (Ψ, ψ) is an automorphism of π , then

$$\Psi_* \circ \mathcal{V} = \mathcal{V} \circ \Psi_*, \quad \Psi_* \circ \mathcal{H} = \mathcal{H} \circ \Psi_*. \quad (3.7)$$

In particular, Ψ_ preserves HM and VM .*

Conversely, if $\Psi : M \rightarrow M$ and $\psi : B \rightarrow B$ are diffeomorphisms satisfying $\pi \circ \Psi = \psi \circ \pi$, then (Ψ, ψ) is an automorphism of π if $\psi^ \check{g} = \check{g}$, (3.7) holds, and $g(\Psi_* v, \Psi_* v) = g(v, v)$ for all $v \in VM$.*

Lemma 3.4. *Let $\pi : (M, g) \rightarrow (B, \check{g})$ be a Riemannian submersion and (Ψ, ψ) an automorphism of π . Then, the norm of O'Neill tensor for π satisfies $\Psi^*|A| = |A|$.*

Proof. Let X_1, X_2 be horizontal vector fields on M . Set $Y_1 = \Psi_* X_1, Y_2 = \Psi_* X_2$. The first relation in (3.7) implies

$$\begin{aligned} 2A_{Y_1} Y_2 &= 2A_{\Psi_* X_1} \Psi_* X_2 = \mathcal{V}[\Psi_* X_1, \Psi_* X_2] = \mathcal{V}(\Psi_*[X_1, X_2]) = \Psi_*(\mathcal{V}[X_1, X_2]) \\ &= 2\Psi_*(A_{X_1} X_2). \end{aligned}$$

At points $x, y \in M$ such that $y = \Psi(x)$, it follows

$$\begin{aligned} g_y(A_{Y_1} Y_2, A_{Y_1} Y_2) &= g_y(\Psi_*(A_{X_1} X_2), \Psi_*(A_{X_1} X_2)) \\ &= g_x(A_{X_1} X_2, A_{X_1} X_2). \end{aligned} \quad (3.8)$$

Let $x \in M$ and $y = \Psi(x)$. We claim $|A|^2(x) = |A|^2(y)$. Take horizontal vector fields E_1, \dots, E_m on M such that $E_1(x), \dots, E_m(x)$ is an orthonormal basis of HM at x . Define horizontal vector fields $F_1 = \Psi_* E_1, \dots, F_m = \Psi_* E_m$. Since $F_1(y), \dots, F_m(y)$ is an orthonormal basis of HM at y , (3.8) implies

$$|A|^2(x) = \sum_{i,j=1}^m g_x(A_{E_i} E_j, A_{E_i} E_j) = \sum_{i,j=1}^m g_y(A_{F_i} F_j, A_{F_i} F_j) = |A|^2(y).$$

This proves $\Psi^*|A| = |A|$. □

3.2 Proof of Theorem 1.2

Let $E \rightarrow B$ be a real vector bundle of rank $k+1 \geq 1$, $\langle \cdot, \cdot \rangle$ a fiberwise inner product on E , and $UE = \{s \in E \mid \langle s, s \rangle = 1\}$ the unit sphere bundle of $(E, \langle \cdot, \cdot \rangle)$. With the restriction map $\pi : UE \rightarrow B$ of the projection $E \rightarrow B$, UE is a S^k -bundle with the orthogonal group as structure group:

$$O(k+1) \curvearrowright S^k \rightarrow UE \xrightarrow{\pi} B.$$

Let \check{g} be a Riemannian metric on B and ∇ a metric connection on $(E, \langle \cdot, \cdot \rangle)$. We define the connection metric $g = \check{g} \oplus \overset{\circ}{g}_k$ on UE with respect to the Ehresmann connection induced by ∇ . With the metric g , π is a Riemannian submersion with totally geodesic fibers:

$$O(k+1) \overset{\text{isom}}{\curvearrowright} S^k(1) \rightarrow (UE, g) \xrightarrow{\pi} (B, \check{g}).$$

Define the fiberwise symmetric bilinear form ξ on the vector bundle E by

$$\xi(s_1, s_2) = \sum_{i,j=1}^m \langle R^\nabla(\check{E}_i, \check{E}_j)s_1, R^\nabla(\check{E}_i, \check{E}_j)s_2 \rangle \quad (3.9)$$

for $s_1, s_2 \in E$. Here, R^∇ is the curvature of ∇ , $m = \dim B$, and $\check{E}_1, \dots, \check{E}_m$ is an orthonormal basis for (B, \check{g}) . ξ does not depend on the choice of $\check{E}_1, \dots, \check{E}_m$. $\xi = 0$ if and only if ∇ is flat. The following formula appears in [12, p. 279]. See also [10], [9, §9] for proof.

Lemma 3.5. *The norm $|A|$ of O'Neill tensor for π satisfies*

$$|A|^2(s) = \frac{1}{4}\xi(s, s) \quad (3.10)$$

for every $s \in UE$.

Proof of Throrem 1.2. Let $(B, \check{g}) = G/H$ be a Riemannian homogeneous space. For an irreducible orthogonal representation $\phi : H \rightarrow \text{SO}(k+1)$ of H on \mathbb{R}^{k+1} , define the vector bundle $E = G \times_\phi \mathbb{R}^{k+1} \rightarrow B$ with fiberwise inner product $\langle \cdot, \cdot \rangle$. The action of G on B lifts to E , and $\langle \cdot, \cdot \rangle$ is invariant under this action. That is,

$$\langle a_*s_1, a_*s_2 \rangle = \langle s_1, s_2 \rangle \quad (3.11)$$

for $a \in G$. In particular, the lifted action of G on E preserves UE .

Let ω be a G -invariant principal H -connection on $G \rightarrow B$. The associated metric connection ∇ on E is invariant under the action of G . The curvature R^∇ is thus invariant under G . That is,

$$a_* (R^\nabla(\check{X}, \check{Y})s) = R^\nabla(a_*\check{X}, a_*\check{Y})a_*s \quad (3.12)$$

for $a \in G$.

Define ξ by (3.9). Let V be the fiber of E over the coset $o = eH$, where e is the identity element of G . Denote by ξ_o and $\langle \cdot, \cdot \rangle_o$ the restrictions of ξ and $\langle \cdot, \cdot \rangle$ to V ,

respectively. ξ_o is invariant under the restricted action of H on V . Indeed,

$$\begin{aligned}
\xi_o(a_*s_1, a_*s_2) &= \sum_{i,j=1}^m \langle R^\nabla(\check{E}_i, \check{E}_j)_{a_*s_1}, R^\nabla(\check{E}_i, \check{E}_j)_{a_*s_2} \rangle_o \\
&= \sum_{i,j=1}^m \langle a_* (R^\nabla(a_*^{-1}\check{E}_i, a_*^{-1}\check{E}_j)s_1), a_* (R^\nabla(a_*^{-1}\check{E}_i, a_*^{-1}\check{E}_j)s_2) \rangle_o \quad \because (3.12) \\
&= \sum_{i,j=1}^m \langle R^\nabla(a_*^{-1}\check{E}_i, a_*^{-1}\check{E}_j)s_1, R^\nabla(a_*^{-1}\check{E}_i, a_*^{-1}\check{E}_j)s_2 \rangle_o \quad \because (3.11) \\
&= \sum_{i,j=1}^m \langle R^\nabla(\check{E}_i, \check{E}_j)s_1, R^\nabla(\check{E}_i, \check{E}_j)s_2 \rangle_o = \xi_o(s_1, s_2)
\end{aligned}$$

for $a \in H$ since ξ is independent of the orthonormal bases at o chosen. Since the representation $H \rightarrow \text{SO}(V)$ is irreducible, it follows from Schur's lemma that ξ_o and $\langle \cdot, \cdot \rangle_o$ are proportional.

Let $g = \check{g} \oplus \check{g}_k^\circ$ be the connection metric on UE with respect to the Ehresmann connection induced by ∇ . Lemma 3.5 implies that $|A|$, the norm of O'Neill tensor for $\pi : (UE, g) \rightarrow (B, \check{g})$, is constant on $\pi^{-1}(o) = UE \cap V$. On the other hand, it follows from Lemma 3.3 that G acts on UE by automorphisms of the Riemannian submersion $(UE, g) \rightarrow (B, \check{g})$. Since G acts on UE fiber-transitively, we conclude from Lemma 3.4 that $|A|$ is constant on UE .

Set $a := |A|$. The O'Neill formula (3.4) implies that the connection metric $g(r) = \check{g} \oplus r^2 \check{g}_k^\circ$ for each $r > 0$, with respect to the same Ehresmann connection as g , has constant scalar curvature $R(\check{g}) + k(k-1)/r^2 - a^2 r^2$. By construction, the projection $(UE, g(r)) \rightarrow (B, \check{g})$ is a Riemannian submersion with totally geodesic fibers, and the typical fiber is isometric to the round sphere $S^k(r)$ (cf. Sect. 2). \square

3.3 Fiberwise join of sphere bundles

For $i = 1, 2$, let S^{k_i} be the sphere of dimension k_i and $\text{O}(k_i + 1) \curvearrowright S^{k_i}$ be the transitive action ($k_1, k_2 \geq 0$). We introduce the differential structure on their join

$$S^{k_1} * S^{k_2} = \frac{S^{k_1} \times [0, T] \times S^{k_2}}{0 \times S^{k_1}, T \times S^{k_2}}$$

through polar coordinates. The action $\text{O}(k_1 + 1) \times \text{O}(k_2 + 1) \curvearrowright S^{k_1} \times [0, T] \times S^{k_2}$ descends smoothly to $S^{k_1} * S^{k_2}$, and $S^{k_1} * S^{k_2}$ is equivariantly diffeomorphic to $S^{k_1+k_2+1}$. Let

$$\hat{\rho} : S^{k_1} \times (0, T) \times S^{k_2} \rightarrow S^{k_1} * S^{k_2} \quad (3.13)$$

be the composition map $S^{k_1} \times (0, T) \times S^{k_2} \hookrightarrow S^{k_1} \times [0, T] \times S^{k_2} \twoheadrightarrow S^{k_1} * S^{k_2}$. $\hat{\rho}$ is an $\text{O}(k_1 + 1) \times \text{O}(k_2 + 1)$ -equivariant injective local diffeomorphism whose image is dense in $S^{k_1} * S^{k_2}$. The following is well-known (cf. [17, pp. 213–214], [4, 4.6], [25, 3.4–4.1]).

Lemma 3.6. *Let $f_i : (0, T) \rightarrow \mathbb{R}$ be a strictly positive C^∞ function for $i = 1, 2$, and consider the doubly warped product metric $f_1^2(t)\overset{\circ}{g}_{k_1} + dt^2 + f_2^2(t)\overset{\circ}{g}_{k_2}$ on $S^{k_1} \times (0, T) \times S^{k_2}$. The scalar curvature of this metric is equal to*

$$-2k_1 \frac{f_1''}{f_1} + k_1(k_1 - 1) \frac{1 - (f_1')^2}{f_1^2} - 2k_2 \frac{f_2''}{f_2} + k_2(k_2 - 1) \frac{1 - (f_2')^2}{f_2^2} - 2k_1 k_2 \frac{f_1' f_2'}{f_1 f_2} \quad (3.14)$$

on $S^{k_1} \times (0, T) \times S^{k_2}$.

*This metric is the pullback by $\hat{\rho}$ of an $O(k_1 + 1) \times O(k_2 + 1)$ -invariant C^∞ Riemannian metric on $S^{k_1} * S^{k_2}$ if and only if f_1, f_2 extend to C^∞ functions on $[0, T]$ and satisfy*

$$\begin{aligned} f_1(0) &> 0, \quad f_1^{(2l-1)}(0) = 0, & f_1(T) &= 0, \quad f_1'(T) = -1, \quad f_1^{(2l)}(T) = 0, \\ f_2(T) &> 0, \quad f_2^{(2l-1)}(T) = 0, & f_2(0) &= 0, \quad f_2'(0) = 1, \quad f_2^{(2l)}(0) = 0 \end{aligned} \quad (3.15)$$

for every $l \geq 1$.

Let $E_i \rightarrow B$ be a real vector bundle of rank $k_i + 1 \geq 1$, $\langle \cdot, \cdot \rangle_i$ a fiberwise inner product on E_i , and ∇^i a metric connection on $(E_i, \langle \cdot, \cdot \rangle_i)$ for $i = 1, 2$. For a Riemannian metric \check{g} on B , define the connection metric $g_i = \check{g} \oplus \overset{\circ}{g}_{k_i}$ on the unit sphere bundle $\pi_i : UE_i \rightarrow B$ with respect to the Ehresmann connection induced by ∇^i as in Sect. 3.2. We introduce the fiberwise join of $\pi_1 : (UE, g_1) \rightarrow (B, \check{g})$ and $\pi_2 : (UE, g_2) \rightarrow (B, \check{g})$ as follows.

Take the direct sum connection $\nabla^1 \oplus \nabla^2$ on $E_1 \oplus E_2$ and consider the induced Ehresmann connection HM on the unit sphere bundle $\pi : M = U(E_1 \oplus E_2) \rightarrow B$. For an $O(k_1 + 1) \times O(k_2 + 1)$ -invariant metric \hat{g} on $S^{k_1+k_2+1} \cong S^{k_1} * S^{k_2}$, define the connection metric $g = \check{g} \oplus \hat{g}$ on $U(E_1 \oplus E_2)$ with respect to HM . The fiberwise join of π_1 and π_2 with respect to \hat{g} is the Riemannian submersion $\pi : (U(E_1 \oplus E_2), g) \rightarrow (B, \check{g})$ with totally geodesic fibers. Summarizing the notations,

$$O(k_1 + 1) \times O(k_2 + 1) \overset{\text{isom}}{\curvearrowright} (S^{k_1} * S^{k_2}, \hat{g}) \rightarrow (U(E_1 \oplus E_2), g) \xrightarrow{\pi} (B, \check{g}).$$

Equivalently, we can describe the fiberwise join in terms of principal bundles. Consider (1) the orthonormal frame bundle $P_i \rightarrow B$ of $(E_i, \langle \cdot, \cdot \rangle_i)$ and principal connection ω_i associated with ∇^i , (2) their fiberwise product $P = P_1 \times_B P_2 \rightarrow B$ and $\omega = \omega_1 \times_B \omega_2$, and (3) the bundle $UE_1 *_B UE_2 := P \times_{O(k_1+1) \times O(k_2+1)} (S^{k_1} * S^{k_2}) \rightarrow B$ and Ehresmann connection associated with (P, ω) . The unit sphere bundle $\pi : U(E_1 \oplus E_2) \rightarrow B$ with HM is isomorphic to $UE_1 *_B UE_2 \rightarrow B$ with this Ehresmann connection. Symbolically,

$$M = U(E_1 \oplus E_2) \cong UE_1 *_B UE_2.$$

Lemma 3.7. *The equivariant map $\hat{\rho} : S^{k_1} \times (0, T) \times S^{k_2} \rightarrow S^{k_1} * S^{k_2}$ induces a map*

$$\rho : (0, T) \times \left(UE_1 \times_B UE_2 \right) \rightarrow UE_1 *_B UE_2.$$

*ρ is an injective local diffeomorphism whose image is dense in $UE_1 *_B UE_2$.*

Proof. Define ρ so that the diagram

$$\begin{array}{ccc} P \times S^{k_1} \times (0, T) \times S^{k_2} & \xrightarrow{\text{id} \times \hat{\rho}} & P \times (S^{k_1} * S^{k_2}) \\ \downarrow & & \downarrow \\ (0, T) \times \left(U E_1 \times_B U E_2 \right) & \xrightarrow{\rho} & U E_1 *_B U E_2 \end{array}$$

commutes. Here, $U E_1 \times_B U E_2 = P \times_{O(k_1+1) \times O(k_2+1)} (S^{k_1} \times S^{k_2})$ is the fiberwise product. ρ is well-defined since $\hat{\rho}$ is $O(k_1+1) \times O(k_2+1)$ -equivariant. Here, the vertical arrows are quotient maps with respect to the diagonal actions of $O(k_1+1) \times O(k_2+1)$. Note that the associated bundle $P \times_{O(k_1+1) \times O(k_2+1)} (S^{k_1} \times (0, T) \times S^{k_2})$ is canonically isomorphic to $(0, T) \times \left(U E_1 \times_B U E_2 \right)$, for $O(k_1+1) \times O(k_2+1)$ acts on $(0, T)$ trivially. Since $\hat{\rho}$ is an injective local diffeomorphism with dense image, so is ρ . \square

Lemma 3.8. *Let \hat{g} be an $O(k_1+1) \times O(k_2+1)$ -invariant metric on $S^{k_1} * S^{k_2}$ such that $\hat{\rho}^* \hat{g} = f_1^2(t) \overset{\circ}{g}_{k_1} + dt^2 + f_2^2(t) \overset{\circ}{g}_{k_2}$ on $S^{k_1} \times (0, T) \times S^{k_2}$. Consider the fiberwise join $\pi : \left(U E_1 *_B U E_2, g \right) \rightarrow (B, \check{g})$ with respect to \hat{g} .*

Then, the pullback metric $\rho^ g$ on $(0, T) \times \left(U E_1 \times_B U E_2 \right)$ is equal to the connection metric $\check{g} \oplus \left(f_1^2(t) \overset{\circ}{g}_{k_1} + dt^2 + f_2^2(t) \overset{\circ}{g}_{k_2} \right)$ with respect to the Ehresmann associated with ω . Consequently, the scalar curvature of $\rho^* g$ satisfies*

$$\begin{aligned} R(\rho^* g) - (\pi \circ \rho)^* R(\check{g}) &= -2k_1 \frac{f_1''}{f_1} + k_1(k_1 - 1) \frac{1 - (f_1')^2}{f_1^2} - |A^1|^2 f_1^2 \\ &\quad - 2k_2 \frac{f_2''}{f_2} + k_2(k_2 - 1) \frac{1 - (f_2')^2}{f_2^2} - |A^2|^2 f_2^2 - 2k_1 k_2 \frac{f_1' f_2'}{f_1 f_2} \end{aligned} \quad (3.16)$$

on $(0, T) \times \left(U E_1 \times_B U E_2 \right)$. Here, $|A^i|$ is the norm of O'Neill tensor for π_i .

Proof. $\rho^* \check{g} = \check{g} \oplus \left(f_1^2(t) \overset{\circ}{g}_{k_1} + dt^2 + f_2^2(t) \overset{\circ}{g}_{k_2} \right)$ since the map ρ preserves the horizontal and vertical bundles as well as the lengths of horizontal and vertical vectors, respectively.

We prove (3.16). Let $|A|$ be the norm of O'Neill tensor for the Riemannian submersion $\pi \circ \rho : (0, T) \times \left(U E_1 \times_B U E_2 \right) \rightarrow B$. In view of the O'Neill formula (3.1) for scalar curvature and (3.14), we have only to show

$$|A|^2 = |A^1|^2 f_1^2 + |A^2|^2 f_2^2. \quad (3.17)$$

Take basic vector fields X, Y on $(0, T) \times \left(UE_1 \times_{\substack{B}} UE_2\right)$. Since $A_X Y = A_X^1 Y + A_X^2 Y$, $A_X^1 Y \perp A_X^2 Y$, and $A_X^i Y$ is tangent to the fibers of $\pi_i : UE_i \rightarrow B$, there holds

$$\begin{aligned} (\rho^* g)(A_X Y, A_X Y) &= \left(f_1^2 \circ g_{k_1} + dt^2 + f_2^2 \circ g_{k_2}\right)(A_X Y, A_X Y) \\ &= f_1^2 \circ g_{k_1}(A_X^1 Y, A_X^1 Y) + f_2^2 \circ g_{k_2}(A_X^2 Y, A_X^2 Y). \end{aligned}$$

Hence, taking an orthonormal vector field E_1, \dots, E_m to be basic, we obtain

$$\begin{aligned} |A|^2 &= \sum_{i,j=1}^m (\rho^* g)(A_{E_i} E_j, A_{E_i} E_j) \\ &= \sum_{i,j=1}^m f_1^2 \circ g_{k_1}(A_{E_i}^1 E_j, A_{E_i}^1 E_j) + \sum_{i,j=1}^m f_2^2 \circ g_{k_2}(A_{E_i}^2 E_j, A_{E_i}^2 E_j) \\ &= |A^1|^2 f_1^2 + |A^2|^2 f_2^2. \end{aligned}$$

This proves (3.17). \square

Let $\text{cn}_{\mathbf{k}}(t), \text{sn}_{\mathbf{k}}(t)$ be the Jacobi elliptic functions ([32], [31]) and $4K(\mathbf{k})$ their fundamental period ($0 \leq \mathbf{k} < 1$). $\text{cn}_{\mathbf{k}}, \text{sn}_{\mathbf{k}} : \mathbb{R} \rightarrow \mathbb{R}$ satisfy $\text{cn}_{\mathbf{k}}^2 + \text{sn}_{\mathbf{k}}^2 = 1$,

$$\begin{aligned} (\text{cn}'_{\mathbf{k}})^2 &= (1 - \text{cn}_{\mathbf{k}}^2)(1 - \mathbf{k}^2 + \mathbf{k}^2 \text{cn}_{\mathbf{k}}^2), & \text{cn}_{\mathbf{k}}'' &= -2\mathbf{k}^2 \text{cn}_{\mathbf{k}}^3 - (1 - 2\mathbf{k}^2) \text{cn}_{\mathbf{k}}, \\ (\text{sn}'_{\mathbf{k}})^2 &= (1 - \text{sn}_{\mathbf{k}}^2)(1 - \mathbf{k}^2 \text{sn}_{\mathbf{k}}^2), & \text{sn}_{\mathbf{k}}'' &= 2\mathbf{k}^2 \text{sn}_{\mathbf{k}}^3 - (1 + \mathbf{k}^2) \text{sn}_{\mathbf{k}} \end{aligned}$$

on \mathbb{R} . $\text{cn}_{\mathbf{k}}/\sqrt{1 - \mathbf{k}^2}, \text{sn}_{\mathbf{k}}$ are strictly positive on $(0, T)$ and satisfy the boundary conditions (3.15) for f_1, f_2 with $T = K(\mathbf{k})$.

Lemma 3.9. Assume $\check{R}, a_i \geq 0$, and $k_i \geq 1$ are constant for $i = 1, 2$. Suppose $\gamma \in (0, \infty)$ and $\mathbf{k} \in [0, 1)$ satisfy

$$(k_1 + k_2)(k_1 + k_2 + 3)\gamma^4 \mathbf{k}^2(1 - \mathbf{k}^2) = a_1^2 - (1 - \mathbf{k}^2)a_2^2. \quad (3.18)$$

Then, $f_1(t) = \text{cn}_{\mathbf{k}}(\gamma t)/(\gamma\sqrt{1 - \mathbf{k}^2})$ and $f_2(t) = \text{sn}_{\mathbf{k}}(\gamma t)/\gamma$ solve

$$\begin{aligned} R - \check{R} &= -2k_1 \frac{f_1''}{f_1} + k_1(k_1 - 1) \frac{1 - (f_1')^2}{f_1^2} - a_1^2 f_1^2 \\ &\quad - 2k_2 \frac{f_2''}{f_2} + k_2(k_2 - 1) \frac{1 - (f_2')^2}{f_2^2} - a_2^2 f_2^2 - 2k_1 k_2 \frac{f_1' f_2'}{f_1 f_2} \end{aligned} \quad (3.19)$$

on $(0, T)$, are strictly positive on $(0, T)$, and satisfy the boundary conditions (3.15) with $T = K(\mathbf{k})/\gamma$,

$$R - \check{R} = -2(k_1 + k_2)(k_1 + 1)\gamma^2 \mathbf{k}^2 + (k_1 + k_2)(k_1 + k_2 + 1)\gamma^2 - a_2^2/\gamma^2. \quad (3.20)$$

Proof. For arbitrary $\gamma \in (0, \infty)$ and $\mathbf{k} \in [0, 1)$, $f_1(t) = \text{cn}_{\mathbf{k}}(\gamma t)/(\gamma\sqrt{1-\mathbf{k}^2})$ and $f_2(t) = \text{sn}_{\mathbf{k}}(\gamma t)/\gamma$ are strictly positive on $(0, T)$ and satisfy the boundary conditions (3.15) with $T = K(\mathbf{k})/\gamma$. Since

$$\begin{aligned}(f_1'(t))^2 &= \left(\frac{\text{cn}'_{\mathbf{k}}(\gamma t)}{\sqrt{1-\mathbf{k}^2}} \right)^2 = -\frac{\mathbf{k}^2}{1-\mathbf{k}^2} \text{cn}_{\mathbf{k}}^4(\gamma t) - \frac{1-2\mathbf{k}^2}{1-\mathbf{k}^2} \text{cn}_{\mathbf{k}}^2(\gamma t) + 1 \\ &= -\gamma^4 \mathbf{k}^2 (1-\mathbf{k})^2 f_1^4(t) - \gamma^2 (1-2\mathbf{k}^2) f_1^2(t) + 1, \\ f_1''(t) &= -2\gamma^4 \mathbf{k}^2 (1-\mathbf{k})^2 f_1^3(t) - \gamma^2 (1-2\mathbf{k}^2) f_1(t),\end{aligned}$$

$$-\frac{f_1''}{f_1} = 2\gamma^4 \mathbf{k}^2 (1-\mathbf{k})^2 f_1^2 + \gamma^2 (1-2\mathbf{k}^2), \quad \frac{1-(f_1')^2}{f_1^2} = \gamma^4 \mathbf{k}^2 (1-\mathbf{k})^2 f_1^2 + \gamma^2 (1-2\mathbf{k}^2),$$

we obtain

$$-2k_1 \frac{f_1''}{f_1} + k_1(k_1-1) \frac{1-(f_1')^2}{f_1^2} = k_1(k_1+3) \gamma^4 \mathbf{k}^2 (1-\mathbf{k}^2) f_1^2 + k_1(k_1+1) \gamma^2 (1-2\mathbf{k}^2). \quad (3.21)$$

Similarly, since

$$\begin{aligned}(f_2'(t))^2 &= (\text{sn}'_{\mathbf{k}}(\gamma t))^2 = \mathbf{k}^2 \text{sn}_{\mathbf{k}}^4(\gamma t) - (1+\mathbf{k}^2) \text{sn}_{\mathbf{k}}^2(\gamma t) + 1 \\ &= \gamma^4 \mathbf{k}^2 f_2^4(t) - \gamma^2 (1+\mathbf{k}^2) f_2^2(t) + 1, \\ f_2''(t) &= 2\gamma^4 \mathbf{k}^2 f_2^3(t) - \gamma^2 (1+\mathbf{k}^2) f_2(t),\end{aligned}$$

$$-\frac{f_2''}{f_2} = -2\gamma^4 \mathbf{k}^2 f_2^2 + \gamma^2 (1+\mathbf{k}^2), \quad \frac{1-(f_2')^2}{f_2^2} = -\gamma^4 \mathbf{k}^2 f_2^2 + \gamma^2 (1+\mathbf{k}^2),$$

we obtain

$$-2k_2 \frac{f_2''}{f_2} + k_2(k_2-1) \frac{1-(f_2')^2}{f_2^2} = -k_2(k_2+3) \gamma^4 \mathbf{k}^2 f_2^2 + k_2(k_2+1) \gamma^2 (1+\mathbf{k}^2). \quad (3.22)$$

The relation $\text{cn}_{\mathbf{k}}^2 + \text{sn}_{\mathbf{k}}^2 = 1$ yields

$$(1-\mathbf{k}^2) f_1^2 + f_2^2 = 1/\gamma^2, \quad (1-\mathbf{k}^2) f_1 f_1' + f_2 f_2' = 0, \quad (3.23)$$

$$-2k_1 k_2 \frac{f_1' f_2'}{f_1 f_2} = 2k_1 k_2 \frac{f_2'}{f_1^2 f_2} \frac{f_2 f_2'}{1-\mathbf{k}^2} = \frac{2k_1 k_2}{1-\mathbf{k}^2} \frac{(f_2')^2}{f_1^2}.$$

Since we can also write

$$\begin{aligned}(f_2'(t))^2 &= (\text{sn}'_{\mathbf{k}}(\gamma t))^2 = (1-\text{sn}_{\mathbf{k}}^2(\gamma t))(1-\mathbf{k}^2 \text{sn}_{\mathbf{k}}^2(\gamma t)) \\ &= \gamma^2 (1-\mathbf{k}^2) f_1^2(t) (1-\gamma^2 \mathbf{k}^2 f_2^2(t)),\end{aligned}$$

there holds

$$-2k_1k_2\frac{f_1'f_2'}{f_1f_2} = -2k_1k_2\gamma^4\mathbf{k}^2f_2^2 + 2k_1k_2\gamma^2. \quad (3.24)$$

Hence (3.22), (3.21), and (3.24) imply

$$\begin{aligned} & -2k_1\frac{f_1''}{f_1} + k_1(k_1-1)\frac{1-(f_1')^2}{f_1^2} - 2k_2\frac{f_2''}{f_2} + k_2(k_2-1)\frac{1-(f_2')^2}{f_2^2} - 2k_1k_2\frac{f_1'f_2'}{f_1f_2} \\ &= k_1(k_1+3)\gamma^4\mathbf{k}^2(1-\mathbf{k}^2)f_1^2 + k_1(k_1+1)\gamma^2(1-2\mathbf{k}^2) \\ & \quad - k_2(k_2+3)\gamma^4\mathbf{k}^2f_2^2 + k_2(k_2+1)\gamma^2(1+\mathbf{k}^2) \\ & \quad - 2k_1k_2\gamma^4\mathbf{k}^2f_2^2 + 2k_1k_2\gamma^2 \\ &= k_1(k_1+3)\gamma^4\mathbf{k}^2(1-\mathbf{k}^2)f_1^2 - k_2(2k_1+k_2+3)\gamma^4\mathbf{k}^2f_2^2 \\ & \quad + (k_1+k_2)(k_1+k_2+1)\gamma^2 - (2k_1^2-k_2^2+2k_1-k_2)\gamma^2\mathbf{k}^2. \end{aligned}$$

Under the assumption (3.18),

$$1-\mathbf{k}^2 : 1 = k_1(k_1+3)\gamma^4\mathbf{k}^2(1-\mathbf{k}^2) - a_1^2 : -(k_2(2k_1+k_2+3)\gamma^4\mathbf{k}^2 + a_2^2)$$

holds so that

$$\begin{aligned} & -2k_1\frac{f_1''}{f_1} + k_1(k_1-1)\frac{1-(f_1')^2}{f_1^2} - a_1^2f_1^2 \\ & - 2k_2\frac{f_2''}{f_2} + k_2(k_2-1)\frac{1-(f_2')^2}{f_2^2} - a_2^2f_2^2 - 2k_1k_2\frac{f_1'f_2'}{f_1f_2} \\ &= (k_1(k_1+3)\gamma^4\mathbf{k}^2(1-\mathbf{k}^2) - a_1^2)f_1^2 - (k_2(2k_1+k_2+3)\gamma^4\mathbf{k}^2 + a_2^2)f_2^2 \\ & \quad + (k_1+k_2)(k_1+k_2+1)\gamma^2 - (2k_1^2-k_2^2+2k_1-k_2)\gamma^2\mathbf{k}^2 \\ &= -(k_2(2k_1+k_2+3)\gamma^4\mathbf{k}^2 + a_2^2)/\gamma^2 \\ & \quad + (k_1+k_2)(k_1+k_2+1)\gamma^2 - (2k_1^2-k_2^2+2k_1-k_2)\gamma^2\mathbf{k}^2 \\ &= -(2k_1k_2+2k_1^2+2k_2+2k_1)\gamma^2\mathbf{k}^2 + (k_1+k_2)(k_1+k_2+1)\gamma^2 - a_2^2/\gamma^2 \\ &= -2(k_1+k_2)(k_1+1)\gamma^2\mathbf{k}^2 + (k_1+k_2)(k_1+k_2+1)\gamma^2 - a_2^2/\gamma^2 \\ &= R - \check{R}. \end{aligned}$$

This proves (3.19). \square

Proof of Theorem 1.5. Note that the norm $|A^i|$ of O'Neill tensor for $\pi_i : (UE_i, g_i) \rightarrow (B, \check{g})$ is constant since g_i and \check{g} have constant scalar curvature (cf. Proposition 3.2). $|A^i| = 0$ if and only if ∇^i is flat.

Let $\check{R} = R(\check{g})$, $a_i = |A^i|$. Suppose $\gamma \in (0, \infty)$ and $\mathbf{k} \in [0, 1)$ satisfy (3.18), and set $R \in \mathbb{R}$ by (3.20). Define $T = K(\mathbf{k})/\gamma$, $f_1(t) = \text{cn}_{\mathbf{k}}(\gamma t)/(\gamma\sqrt{1-\mathbf{k}^2})$, and $f_2(t) = \text{sn}_{\mathbf{k}}(\gamma t)/\gamma$. Since f_1 and f_2 are positive and satisfy the boundary conditions (Lemma 3.9), there exists an $O(k_1+1) \times O(k_2+1)$ -invariant C^∞ Riemannian metric \hat{g} on $S^{k_1} * S^{k_2}$ such that $\hat{\rho}^*\hat{g} = f_1^2(t)\overset{\circ}{g}_{k_1} + dt^2 + f_2^2(t)\overset{\circ}{g}_{k_2}$ on $S^{k_1} \times (0, T) \times S^{k_2}$ (Lemma 3.8).

Let $\pi : (U(E_1 \oplus E_2), g) \rightarrow (B, \check{g})$ be the fiberwise join of $\pi_i : (UE, g_i) \rightarrow (B, \check{g})$ ($i = 1, 2$). Since f_1 and f_2 solve (3.19), it follows from (3.16) that the scalar curvature of ρ^*g is identically equal to R on $(0, T) \times \left(UE_1 \times_B UE_2\right)$. Therefore, g has constant scalar curvature R on $U(E_1 \oplus E_2)$ since $\rho : (0, T) \times \left(UE_1 \times_B UE_2\right) \rightarrow U(E_1 \oplus E_2)$ has dense image (Lemma 3.7).

For the rest of proof, we solve (3.18), (3.20) explicitly. Firstly, observe the following: If $\gamma \in (0, \infty)$, $\mathbf{k} \in (0, 1)$ satisfy (3.18) and if $a_1^2 - (1 - \mathbf{k}^2)a_2^2 > 0$, then

$$\gamma^2 = \sqrt{\frac{1}{(k_1 + k_2)(k_1 + k_2 + 3)}} \sqrt{\frac{a_1^2 - (1 - \mathbf{k}^2)a_2^2}{\mathbf{k}^2(1 - \mathbf{k}^2)}} > 0, \quad (3.25)$$

$$R - \check{R} = (k_1 + k_2)(k_1 + k_2 + 1 - 2(k_1 + 1)\mathbf{k}^2)\gamma^2 - a_2^2/\gamma^2 \quad (3.26)$$

$$\begin{aligned} &= \sqrt{\frac{k_1 + k_2}{k_1 + k_2 + 3}} \sqrt{\frac{a_1^2 - (1 - \mathbf{k}^2)a_2^2}{\mathbf{k}^2(1 - \mathbf{k}^2)}} (k_1 + k_2 + 1 - 2(k_1 + 1)\mathbf{k}^2) \\ &\quad - a_2^2 \sqrt{(k_1 + k_2)(k_1 + k_2 + 3)} \sqrt{\frac{\mathbf{k}^2(1 - \mathbf{k}^2)}{a_1^2 - (1 - \mathbf{k}^2)a_2^2}}. \end{aligned} \quad (3.27)$$

If $a_1 = a_2 = 0$, then we take $\mathbf{k} = 0$ and an arbitrary $\gamma > 0$ so that $R - \check{R} = (k_1 + k_2)(k_1 + k_2 + 1)\gamma^2$. The resulting metrics are locally the direct products of (B, \check{g}) and the $(k_1 + k_2 + 1)$ -dimensional round spheres in this case. Interchanging E_1 and E_2 if necessary, assume $a_1 > 0$.

- Case 1: $a_1 > a_2$. For each $\mathbf{k} \in (0, 1)$, define $\gamma > 0$ by (3.25). As $\mathbf{k} \rightarrow 0$, $\gamma \rightarrow \infty$. Hence $R \rightarrow \infty$ as $\mathbf{k} \rightarrow 0$ by (3.26).
- Case 2: $a_1 = a_2 > 0$. We can take $\mathbf{k} = 0$ and an arbitrary $\gamma > 0$ so that $R - \check{R} = (k_1 + k_2)(k_1 + k_2 + 1)\gamma^2 - a_2^2/\gamma^2$. We can also define $\gamma > 0$ by (3.25) for each $\mathbf{k} \in (0, 1)$. As $\mathbf{k} \rightarrow 0$, $\gamma \rightarrow \infty$. Hence $R \rightarrow \infty$ as $\mathbf{k} \rightarrow 0$ by (3.26).
- Case 3: $0 < a_1 < a_2$. For each $\mathbf{k} \in (1 - a_1^2/a_2^2, 1)$, define $\gamma > 0$ by (3.25). As $\mathbf{k} \rightarrow 1 - a_1^2/a_2^2$, $\gamma \rightarrow 0$. Hence $R \rightarrow -\infty$ as $\mathbf{k} \rightarrow 1 - a_1^2/a_2^2$ by (3.26).

In all these cases, we can take \mathbf{k} arbitrarily close to 1. Lastly, we see the limiting behavior of R as $\mathbf{k} \rightarrow 1$. We claim:

$$\text{If } k_2 > k_1 + 1, \text{ then } R \rightarrow \infty \text{ as } \mathbf{k} \rightarrow 1. \quad (3.28)$$

$$\text{If } k_2 = k_1 + 1, \text{ then } R \rightarrow 0 \text{ as } \mathbf{k} \rightarrow 1. \quad (3.29)$$

$$\text{If } k_2 < k_1 + 1, \text{ then } R \rightarrow -\infty \text{ as } \mathbf{k} \rightarrow 1. \quad (3.30)$$

Observe $\gamma \rightarrow \infty$ as $\mathbf{k} \rightarrow 1$. If $k_2 > k_1 + 1$, then

$$\lim_{\mathbf{k} \rightarrow 1} (k_1 + k_2 + 1 - 2(k_1 + 1)\mathbf{k}^2) = -k_1 + k_2 - 1 > 0$$

holds so that (3.26) implies (3.28). We obtain (3.30) similarly. Suppose $k_2 = k_1 + 1$. Since l'Hôpital's rule yields

$$\lim_{\mathbf{k} \rightarrow 1} \left(\frac{k_1 + k_2 + 1 - 2(k_1 + 1)\mathbf{k}^2}{\sqrt{1 - \mathbf{k}^2}} \right)^2 = \lim_{\mathbf{k} \rightarrow 1} 4(k_1 + 1)(k_1 + k_2 + 1 - 2(k_1 + 1)\mathbf{k}^2) = 0,$$

(3.27) implies (3.29). \square

Remark 3.10. There are one or two one-parameter families of connection metrics with constant scalar curvature on $U(E_1 \oplus E_2) \cong U(E_2 \oplus E_1)$. (1) If both ∇^1 and ∇^2 are flat, then there is a one-parameter family of locally product metrics. (2) Suppose either ∇^1 or ∇^2 is flat but the other one is not flat. Interchange E_1 and E_2 if necessary and assume ∇^2 is flat. In the notation of the previous proof, $a_1 > 0$ and $a_2 = 0$, and the consideration for Case 1 shows that there is a one-parameter family. (3) Suppose neither ∇^1 nor ∇^2 is flat. If $a_1 = a_2$, then there are two one-parameter families (Case 2). If $a_1 \neq a_2$, then, interchanging the roles of E_1 and E_2 , we see that there are two one-parameter families (Case 1 and Case 3).

Remark 3.11. Let $(S^{k_1+k_2+1}, \hat{g}) \rightarrow (U(E_1 \oplus E_2), g) \xrightarrow{\pi} (B, \check{g})$ be the fiberwise join in Theorem 1.5. The preceding proof implies that the following are equivalent: (1) $(S^{k_1+k_2+1}, \hat{g})$ is a round sphere, (2) the scalar curvature of \hat{g} is constant, and (3) the norm $|A|$ of O'Neill tensor for π is constant. (2) and (3) are equivalent since the base space (B, \check{g}) has constant scalar curvature (see Proposition 3.2). (2) implies (1) as follows. \hat{g} has constant scalar curvature if and only if $a_1^2 f_1^2 + a_2^2 f_2^2$ is constant (see (3.14), (3.16)). Since $(1 - \mathbf{k}^2)f_1^2 + f_2^2 = 1/\gamma^2$ holds (see (3.23)), this is possible only if $a_1^2 : a_2^2 = 1 - \mathbf{k}^2 : 1$, so (3.18) yields $\mathbf{k} = 0$. Hence $\text{cn}_{\mathbf{k}} = \cos$ and $\text{sn}_{\mathbf{k}} = \sin$ are trigonometric functions, and \hat{g} has constant sectional curvature.

In particular, if $|A^1| \neq |A^2|$, we cannot rescale the fiber of π to obtain metrics of constant scalar curvature as in the case of Theorem 1.2 (see Proposition 3.2).

3.4 Proof of Theorem 1.4

The proof is based on Theorems 1.2 and 1.5.

Proof of Theorem 1.4. Let $(B, \check{g}) = G/H$ be a Riemannian homogeneous space and $\phi : H \rightarrow \text{SO}(k+1)$ an orthogonal representation of H on \mathbb{R}^{k+1} . We write $\phi = \phi_1 \oplus \phi_2$ where ϕ_i is a trivial or irreducible representation on \mathbb{R}^{k_i+1} for $i = 1, 2$.

If $k_1 + 1 = 0$ or $k_2 + 1 = 0$, then ϕ itself is trivial or irreducible, so we can define connection metrics of constant scalar curvature on $U(G \times_{\phi} \mathbb{R}^{k+1})$ using Theorem 1.2.

Assume $k_1 + 1 \geq 1$, $k_2 + 1 \geq 1$. Firstly, we build connection metrics on $U(G \times_{\phi_i} \mathbb{R}^{k_i+1})$ making use of Theorem 1.2. Then using Theorem 1.5, we define connection metrics of constant scalar curvature on

$$U(G \times_{\phi} \mathbb{R}^{k+1}) = U(G \times_{\phi_1} \mathbb{R}^{k_1+1} \oplus G \times_{\phi_2} \mathbb{R}^{k_2+1}).$$

This proves Theorem 1.4. \square

Every orthogonal representation $\phi : H \rightarrow SO(k+1)$ of a compact group H is completely reducible. Namely, ϕ is equivalent to the direct sum $\mathbf{1}_{k_0+1} \oplus \phi_1 \oplus \cdots \oplus \phi_l$ of the trivial representation $\mathbf{1}_{k_0+1}$ on the Euclidean space of dimension $k_0+1 \geq 0$ and $l \geq 0$ irreducible representations ϕ_1, \dots, ϕ_l of dimension $k_1+1, \dots, k_l+1 \geq 2$.

We cannot apply Theorem 1.5 inductively in general, for the typical fibers do not have constant sectional curvature if $|A^1| \neq |A^2|$ (cf. Remark 3.11). However, provided $l \geq 1$ and $\phi_1 = \cdots = \phi_l$ in the notation above, we can define connection metrics of constant scalar curvature on

$$U(G \times_{\phi} \mathbb{R}^{k+1}) = U\left((B \times \mathbb{R}^{k_0+1}) \oplus G \times_{\phi_1} \mathbb{R}^{k_1+1} \oplus \cdots \oplus G \times_{\phi_l} \mathbb{R}^{k_l+1}\right).$$

For this, we have only to apply Theorem 1.5 for $(l-1)$ or l times according to $k_0+1 = 0$ or $k_0+1 \geq 1$ since $|A^0| = 0$, $|A^1| = \cdots = |A^l|$. It is interesting to ask if Theorem 1.4 holds without any assumption on the orthogonal representation ϕ .

4 The number of constant scalar curvature metrics

Let (M, g) be an n -dimensional closed Riemannian manifold of constant scalar curvature. We consider the Yamabe equation

$$-a_n \Delta_g u + R(g)u = R(g)u^{p_n-1}, \quad u \in C_+^{\infty}(M) \quad (4.1)$$

with $a_n = 4(n-1)/(n-2)$, $p_n = 2n/(n-2)$, and $\Delta = \text{tr } \nabla^2$. A function u solves (4.1) if and only if the conformally deformed metric $u^{p_n-2}g$ has constant scalar curvature $R(g)$. If $R(g) \leq 0$, then the maximum principle implies that only constant functions solve (4.1).

Suppose $H \xrightarrow{\text{isom}} (F, \hat{g}) \xrightarrow{\iota} (M, g) \xrightarrow{\pi} (B, \check{g})$ is a Riemannian submersion with totally geodesic fibers. Let $C^{\infty}(F)^H$ be the subset of $C^{\infty}(F)$ consisting of functions invariant under the action $H \curvearrowright F$. Through parallel transports along horizontal paths, we regard each $\hat{u} \in C^{\infty}(F)^H$ as the function $\iota_* \hat{u}$ on M . More precisely, for $y \in M$, take a horizontal path joining y and a point $x \in F$, and define $(\iota_* \hat{u})(y) = \hat{u}(x)$. $\iota_* \hat{u}$ is well-defined since \hat{u} is invariant under the action of $\text{Hol}(HM)$. The space of all such functions on M is denoted by $\iota_* C^{\infty}(F)^H$. Similarly, $\pi^* C^{\infty}(B)$ denotes the subset of $C^{\infty}(M)$ consisting of functions written as $\pi^* \check{u}$ for some $\check{u} \in C^{\infty}(B)$.

Proposition 4.1. *Let $u \in C^{\infty}(M)$.*

1. *The following three conditions are equivalent. $u \in \pi^* C^{\infty}(B)$. u is constant along each fiber of π . The gradient vector field of u is normal to the fibers of π .*
2. *If $u \in \iota_* C^{\infty}(F)^H$, then the gradient vector field of u is tangent to the fibers of π . Conversely, if the gradient vector field of u is tangent to the fibers of π , then $u \in \iota_* C^{\infty}(F)^{\text{Hol}(HM)}$.*

Proof. We only prove 2. Suppose $u = \iota_* \hat{u}$ for $\hat{u} \in C^\infty(F)^H$. We claim that u is constant along every horizontal path. To show this, let $\gamma : [0, 1] \rightarrow M$ be an arbitrary horizontal path with $\gamma(0) = y_1$ and $\gamma(1) = y_2$. For $i = 1, 2$, take horizontal paths $\gamma_i : [0, 1] \rightarrow M$ with $\gamma_i(0) = x_i$, $\gamma_i(1) = y_i$ for some $x_i \in F$. By definition of $\iota_* \hat{u}$, we have $u(y_i) = \hat{u}(x_i)$. Since $\gamma_2^{-1} \gamma \gamma_1$ is a horizontal path from x_1 to x_2 which projects onto a loop in B , there holds $\hat{u}(x_1) = \hat{u}(x_2)$, for \hat{u} is invariant under $\text{Hol}(HM)$. Hence $u(y_1) = u(y_2)$. This proves the claim. Therefore, as in the proof of Lemma 3.1, we see that $\text{grad } u$ is tangent to the fibers of π .

Conversely, if $\text{grad } u$ is tangent to the fibers of π , then u is constant along the integral curve of every horizontal vector field on M . Hence u is constant along every horizontal path, and $u \in \iota_* C^\infty(F)^{\text{Hol}(HM)}$. \square

Proposition 4.2 (Bérard-Bergery–Bourguignon [3]).

$$\pi^* \circ \Delta_{\check{g}} = \Delta_g \circ \pi^*, \quad \iota_* \circ \Delta_{\hat{g}} = \Delta_g \circ \iota_*. \quad (4.2)$$

Corollary 4.3. *Let $H \xrightarrow{\text{isom}} (F^k, \hat{g}) \xrightarrow{\iota} (M^n, g) \xrightarrow{\pi} (B^m, \check{g})$ be a Riemannian submersion with totally geodesic fibers, and assume g has constant scalar curvature.*

- *Suppose $\check{u} > 0$ is a smooth function on B . Then, $u = \pi^* \check{u}$ solves the Yamabe equation (4.1) on M if and only if \check{u} solves*

$$-a_n \Delta_{\check{g}} \check{u} + R(g) \check{u} = R(g) \check{u}^{p_n-1} \quad \text{on } B. \quad (4.3)$$

- *Suppose $\hat{u} > 0$ is an H -invariant smooth function on F . Then, $u = \iota_* \hat{u}$ solves the Yamabe equation (4.1) on M if and only if \hat{u} solves*

$$-a_n \Delta_{\hat{g}} \hat{u} + R(g) \hat{u} = R(g) \hat{u}^{p_n-1} \quad \text{on } F. \quad (4.4)$$

If $m \geq 1$ and $k \geq 1$, then (4.3) and (4.4) are subcritical in a sense that the respective Sobolev embeddings $W^{1,2} \hookrightarrow L^{p_n}$ are compact.

Let (N, h) be a d -dimensional closed Riemannian manifold. Take constants $N > d$, $R > 0$ and consider the subcritical PDE

$$-a_N \Delta_h v + Rv = Rv^{p_N-1}, \quad v \in C_+^\infty(N). \quad (4.5)$$

The following theorem is due to O. Kobayashi ([18], [19]), Schoen [27] for $d = 1$ and Bidaut-Veron–Veron [8, Theorem 6.1] for $d \geq 2$.

Theorem 4.4. *Assume $d = 1$ and $\lambda_1(-\Delta_h) \geq R/(N-1)$, or $d \geq 2$ and $\text{Ric}(h) \geq \frac{d-1}{d} \frac{R}{N-1} h$. Then $v \equiv 1$ is the unique solution to (4.5).*

Note that $\text{Ric}(h) \geq \frac{d-1}{d} \frac{R}{N-1} h$ implies $\lambda_1(-\Delta_h) \geq R/(N-1)$. O. Kobayashi ([18], [19]), Schoen [27] for $d = 1$ and Petean ([24], see also [16], [14]) for $d \geq 2$ proved the following.

Theorem 4.5. Assume $(N, h) = S^d(r)$ is the round sphere. If $\lambda_l(-\Delta_h) = l(l + d - 1)/r^2 < R/(N - 1)$, then (4.5) has at least $l + 1$ solutions invariant under the cohomogeneity-one action of $\mathrm{SO}(d)$ on S^d .

Corollary 4.3, Theorems 4.4, 4.5 then imply the following.

Proposition 4.6. Let $H \overset{\text{isom}}{\curvearrowright} (F^k, \hat{g}) \xrightarrow{\iota} (M^n, g) \xrightarrow{\pi} (B^m, \check{g})$ be a Riemannian submersion with totally geodesic fibers. Suppose g has constant scalar curvature $R(g) > 0$ and $m, k \geq 1$.

1. If $\mathrm{Ric}(\hat{g}) \geq \frac{k-1}{k} \frac{R(g)}{n-1} \hat{g}$, then the solution $u \in \iota_* C_+^\infty(F)^H$ to (4.1) is unique.
2. If $H \curvearrowright (F, \hat{g}) = \mathrm{SO}(k) \curvearrowright S^k(r)$ and $l(l + k - 1)/r^2 < R(g)/(n - 1)$, then there are at least $l + 1$ solutions to (4.1) in $\iota_* C_+^\infty(F)^H$.
3. If $\mathrm{Ric}(\check{g}) \geq \frac{m-1}{m} \frac{R(g)}{n-1} \check{g}$, then the solution $u \in \pi^* C_+^\infty(B)$ to (4.1) is unique.
4. If $(B^m, \check{g}) = S^m(1)$ and $l(l + m - 1) < R(g)/(n - 1)$, then there are at least $l + 1$ solutions to (4.1) in $\pi^* C_+^\infty(B)$.

The metrics in Theorems 1.2, 1.4 have constant scalar curvature and are defined on the total spaces of Riemannian submersions with totally geodesic fibers. Applying Proposition 4.6, we can study the number of constant scalar curvature metrics in their conformal classes.

Proof of Theorem 1.3. There hold: $(B, \check{g}) = S^m(1)$, $(F, \hat{g}) = S^k(r)$, and $R(g(r)) = m(m - 1) + k(k - 1)/r^2 - a^2 r^2$.

We prove (2) and (3). $\mathrm{Ric}(\check{g}) \geq \frac{m-1}{m} \frac{R(g(r))}{n-1} \check{g}$ is equivalent to $-(a^2/k)r^2 + (k-1)/r^2 \leq m$. $l(l + m - 1) < R(g(r))/(n - 1)$ is equivalent to $-a^2 r^2 + k(k - 1)/r^2 > l(l + m - 1)(m + k - 1) - m(m - 1)$. For a smooth function u on M , $u \in \pi^* C^\infty(B)$ if and only if u is constant along each fiber of π (Proposition 4.1). Hence (3) and (4) in Proposition 4.6 imply (2) and (3). Similarly, (1) of Proposition 4.6 implies (1) of Theorem 1.3. \square

Remark 4.7. Some of the connection metrics that we constructed in Theorems 1.2, 1.4 are positive Yamabe minimizers. Indeed, some of the metrics in Theorems 1.2, 1.4 are scalar flat, and thus they are unique unit-volume metrics of constant scalar curvature in their conformal classes and are strictly stable with respect to the Yamabe functionals. Applying a slight modification of the previous result due to Böhm–Wang–Ziller [11, Theorem 5.1], we see that the connection metrics of constant scalar curvature close to the scalar flat ones are also Yamabe (cf. [22], [23, Proposition 4]).

Remark 4.8. The $(l + 1)$ metrics of constant scalar curvature in (3) of Theorem 1.3 should be non-isometric to each other. Assume $g(r)$ is not conformally flat and $n = m + k \geq 4$ (use the Cotton tensor instead of Weyl tensor if $n = 3$). For $i = 1, 2$, let u_i be a conformal factor such that $\tilde{g}_i := u_i^{p_n-2} g(r)$ has constant scalar curvature $R(g(r))$.

The norm of Weyl tensor satisfies $|W_{\tilde{g}_i}| = u_i^{2-p_n} |W_{g(r)}|$. Since u_i is constant along each fiber and $W_{g(r)}$ is invariant under the fiber-transitive action of G on UE ,

$$\max_{UE} |W_{\tilde{g}_i}| = (\min_B u_i)^{2-p_n} \max_{UE} |W_{g(r)}| > 0.$$

If \tilde{g}_1 and \tilde{g}_2 are isometric to each other, then $\max |W_{\tilde{g}_1}| = \max |W_{\tilde{g}_2}|$ holds so that $\min u_1 = \min u_2$. The u_i 's are obtained as solutions of ODE (see [24]), and one can check that if $\min u_1 = \min u_2$, then $u_1 = u_2$. With this argument, one could see that the metrics in Theorem 1.3 should be non-isometric to each other in most cases. However, it is difficult to rule out the possibility that some of them are isometric.

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